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“NUMERICAL SOLUTION OF THE MULTIPHASE DYNAMIC MODEL OF THE GAS LIFT PROCESS”

ABSTRACT
of the dissertation submitted for the degree of doctor of philosophy (PhD) in the educational program 8D05401 – “Mathematics”

General characteristics of the work. The dissertation is devoted to the development and investigation of a numerical solution for a multiphase dynamic model of the gas-lift process. The study considers both direct and inverse problems described by one-dimensional Navier-Stokes equations for compressible gas. Additionally, numerical methods are developed and analyzed for the two-dimensional case of fluid flow in an axisymmetric domain, allowing for the consideration of complex geometries and spatial distribution of parameters under real well operation conditions.

Relevance of the research. The gas-lift process is widely used in oil production as an effective method to enhance oil recovery. However, optimizing this process-especially in terms of managing the injected gas volume and initial pressure conditions-requires the construction of adequate mathematical models and numerical algorithms. One of the most efficient approaches is the method of inverse problems. Modern methods for solving inverse problems are based on their reduction to optimal control problems, which enables the application of variational analysis and adjoint equation techniques. The relevance of this work lies in the need to develop stable and convergent numerical methods for solving such classes of problems in order to improve the efficiency of managing gas-lift wells.

This dissertation further investigates the Navier-Stokes equations for compressible gas with discontinuous coefficients, describing the gas-lift oil production process. A variational approach is used to solve the inverse problem numerically. The minimization of the objective functional leads to an adjoint problem. A gradient iterative method is applied for the numerical determination of the initial conditions in the direct problem. Numerical calculations were performed to validate the obtained theoretical results.

The dissertation also examines the problem of viscous fluid flow through an L -shaped domain with a prescribed pressure $p = p_{\text{вых}}$ at the outlet and a zero tangential component of the velocity vector at both the inlet and outlet (uniform flow). This problem formulation allows for the determination of the reservoir pressure $p = p_{\text{пл}}$ at the inlet of a production oil well and has significant practical relevance. The solution is based on the Navier–Stokes equations.

The issues of correct formulation of non-standard boundary conditions have been studied. Pressure and tangential velocity component conditions are imposed in the reservoir and at the outlet of the production well.

Research objective. The main objective of the dissertation is to develop and study a numerical method for solving the direct and inverse problems for the Navier–

Stokes equations of compressible gas that describe the gas-lift oil production process. Another important goal is to develop a method for determining the reservoir pressure at the wellbore.

Research tasks:

1) To construct a family of finite difference schemes for solving the direct problem for the Navier–Stokes equations.

2) To investigate the correctness of the discrete model and the stability of the numerical solution.

3) To formulate the inverse problem as an optimal control problem with an additional condition.

4) To derive the adjoint problem based on the Lagrange identity.

5) To develop a gradient method for minimizing the objective functional.

6) To conduct numerical experiments for recovering initial conditions and constructing the performance curve of the gas-lift process.

7) To develop a numerical method for solving the Navier–Stokes equations with non-standard boundary conditions.

Object and subject of the research. The object of the research is the Navier–Stokes equations for compressible gas that describe the gas-lift oil production process.

The subject of the research is numerical methods for solving direct and inverse problems arising in the modeling of this process.

Research methods. The study employs numerical methods for solving inverse problems, variational calculus, optimal control theory, numerical analysis, and the theory of finite difference schemes for partial differential equations. The algorithms were implemented using Python version 3.13.2.

Scientific novelty.

1) A finite difference scheme is proposed for solving the adjoint retrospective problem arising from the minimization of the objective functional in the inverse problem for the Navier–Stokes equations modeling the gas-lift process. The stability and convergence of the scheme are proven.

2) A numerical algorithm based on the gradient method has been developed for the inverse problem of the gas-lift process, where the functional is minimized under additional constraints on the solution and its derivatives at the final time. The adjoint problem is derived using the Lagrange identity and contains valuable information about the direct problem.

3) A connection is established between the solution of the adjoint problem and the gradient of the functional, which enabled the effective implementation of the adjoint equation method in a retrospective formulation.

4) Numerical experiments were conducted to demonstrate the ability to recover the initial conditions (pressure and flow rate) from known values at the final time layer, confirming the correctness of the variational formulation of the inverse problem.

5) The proposed method proved effective in determining the production characteristics of the gas-lift well, which is of great practical importance for the design and operational management of oil fields.

6) A numerical algorithm has been developed for determining the reservoir pressure in a domain with complex geometry (L-shaped domain), based on the transformation of the Navier–Stokes equations into "stream function – vorticity" variables and their integration along the contour. An original method is proposed for calculating the inlet pressure from the known outlet pressure.

Theoretical and practical significance of the research. The results of this dissertation can be used in the development of automated control systems for the gas-lift process, as well as for constructing performance curves and evaluating reservoir pressure. The proposed numerical methods are also applicable to a broader class of optimal control problems governed by partial differential equations.

Main provisions submitted for defense.

– The variational formulation of the inverse problem and the corresponding adjoint problem;

– The algorithm for the numerical implementation of the gradient method with gradient evaluation via the solution of the adjoint problem;

– Stable and convergent finite difference schemes for the Navier–Stokes equations describing the gas-lift process;

– A numerical solution method for the Navier–Stokes equations with non-standard boundary conditions;

– Results of numerical experiments confirming the effectiveness of the proposed method.

Validity and reliability of the scientific provisions, conclusions, and results of the dissertation. The validity of the results is ensured by the rigorous mathematical derivation of the equations, accurate approximation, theoretical analysis of the stability of the numerical schemes, and numerical experiments conducted with controlled accuracy and convergence monitoring.

Research dissemination and validation. The main findings and results of the research were presented and discussed at the following scientific conferences:

– International Conference "Current Problems of Computational and Applied Mathematics" dedicated to the 90th anniversary of academician G.I. Marchuk (Novosibirsk, October 19–23, 2015);

– 5th International Conference "Control and Optimization with Industrial Applications" (Baku, Azerbaijan, August 27–29, 2015);

– International Conference "Computational and Information Technologies in Science, Engineering, and Education" (Almaty, September 24–27, 2015);

– 1st International Scientific and Technical Conference of Students, Master's Students, and Young Scientists "Creativity of Youth for the Innovative Development of Kazakhstan" (Ust-Kamenogorsk, April 9–10, 2015);

– International Conference "Mathematical Sciences and Statistics" (Kuala Lumpur, Malaysia, January 26–28, 2016);

– 3rd International Conference on Analysis and Applied Mathematics (Almaty, September 7–10, 2016);

– Scientific seminar of the Research Institute of Applied Mathematics of Baku State University under the leadership of Academician F.A. Aliev (Baku, Azerbaijan, April 12, 2016);

– Scientific seminars of the Faculty of Information Technology and Business at D. Serikbayev East Kazakhstan Technical University, the Department of Mathematics at S. Amanzholov East Kazakhstan University, and the interuniversity seminar “Mathematical Modeling of Physical and Technogenic Processes” under the leadership of Doctor of Physical and Mathematical Sciences, Professor N.M. Temirbekov (Ust-Kamenogorsk).

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In addition, the results of the study are currently being validated under the grant-funded research project “Zhas Galym” for 2024–2026, implemented under project number IRN AP22683374 titled: “Numerical Solution of a Multiphase Dynamic Model of the Gas-Lift Process”.

Publications and personal contribution of the applicant. In joint works, the scientific advisor was responsible for the formulation of the problem, while the applicant contributed to obtaining theoretical results, conducting numerical experiments, and analyzing the outcomes.

1. Temirbekov N.M., Turarov A.K., Kasenov S.E. // Numerical Solution of the Direct and Inverse Problems in the Gas Lift Process of Oil Production Using the Conjugate Equations Method. Applied System Innovation, 2025, Vol. 8, No. 2. DOI: 10.3390/asi8020047.

2. Aliev F.A., Aliev N.A., Hasanov K.G., Guliev A.P., Turarov A.K., Isaeva G.V. Numerical-analytical method for solving of the first order partial quasi-linear equations. TWMS Journal of Pure and Applied Mathematics, Vol. 6, No. 2, 2015, pp. 158–164.

3. Zhumagulov B.T., Turarov A.K., Temirbekov N.M. Numerical modeling of the gas-lift oil production process using optimization methods. Bulletin of the NAS RK, 2025, Vol. 95, No. 1, pp. 216–236. DOI: 10.47533/2025.1606-146X.22.

4. Temirbekov N.M., Turarov A.K. Numerical solution of a one-dimensional model of the gas-lift process. Proceedings of the National Academy of Sciences of the Republic of Kazakhstan, Physical and Mathematical Series, No. 2 (306), 2016, pp. 159–168.

5. Zhumagulov, N., Temirbekov, N., Baigereyev, D., and Turarov, A. Stability Analysis of a Difference Scheme for the Dynamic Model of Gas Lift Process. In: Computational and Information Technologies in Science, Engineering and Education, 9th International Conference (CITech 2018), Revised Selected Papers, Springer CCIS, Vol. 998, 2018, Ust-Kamenogorsk, Kazakhstan, pp. 236–246.

Structure and volume of the dissertation. The dissertation consists of an introduction, three chapters, a conclusion, a list of references, and appendices. It has a consistent and logically organized structure. The total length of the dissertation is 131 pages.

Main content of the dissertation. Chapter One addresses the theoretical foundations of problem formulation for the Navier–Stokes equations describing the gas-lift process. It presents the physics of gas and gas-liquid mixture flow in the tubing and annular space of an oil well. A mathematical model is provided based on the Navier–Stokes system for both one-dimensional and two-dimensional cases. The direct and inverse problems are formulated. The inverse problem is reduced to an optimal control problem with additional conditions specified at the final time layer. An adjoint retrospective problem is derived using the Lagrange identity.

Chapter Two describes the numerical methods for solving the direct and inverse problems. Stable finite difference schemes are constructed for the direct problem, accounting for the specific features of the differential equations and possible discontinuities in coefficients. An algorithm is developed for the numerical solution of the adjoint problem and for the gradient method used to minimize the objective functional. A method for computing the gradient of the objective functional is presented. The convergence of the iterative method is proven, and the correctness of the calculations is substantiated.

Chapter Three is devoted to the problem of numerically determining the reservoir pressure at the inlet of a production well. The problem is formulated for the Navier–Stokes equations expressed in “stream function – vorticity” variables in a two-dimensional L-shaped domain. Finite difference schemes are constructed to solve the Navier–Stokes equations and to calculate the inlet pressure from known values of the stream function and vorticity. The algorithm for numerical integration of the inlet pressure is based on contour integration. Numerical results are analyzed: pressure and velocity component distributions are obtained, and pressure profiles at the inlet and outlet are plotted. The effectiveness of the algorithm is demonstrated for recovering reservoir pressure and determining the production characteristics of the oil well.